Adapting the Directed Grid Theorem into an FPT algorithm

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Joint work with Ana K. Maia, Ignasi Sau, and Victor Campos.
Work partially done at LIRMM, Montpellier, France.

XP and FPT

Problem with input size n, associated *parameter* k:

- XP problem $\Rightarrow f(k) \cdot n^{g(k)}$ time algorithm.
  - Example: $\mathcal{O}(n^k)$.

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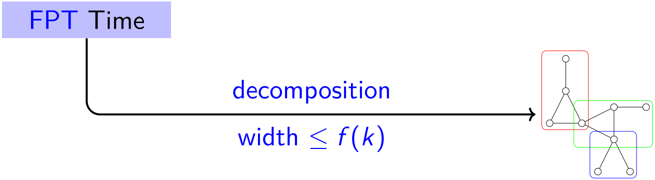
- $\mathrm{W}[1]$-hard problem $\Rightarrow$ strong evidence that it is *not* FPT.

(Undirected) Grid Theorem



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Graph minors V. Excluding a planar graph.
Journal of Combinatorial Theory, Series B, 1986.

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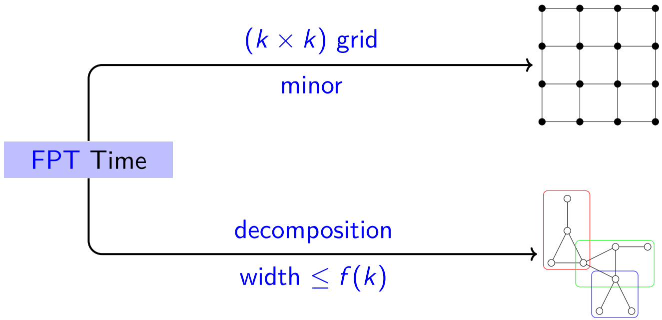

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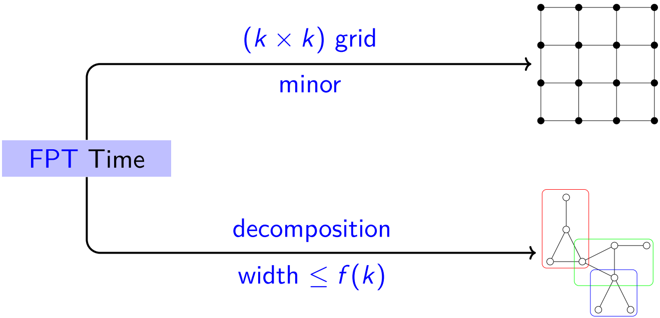

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- Key ingredient in proof of Wagner's Conjecture.

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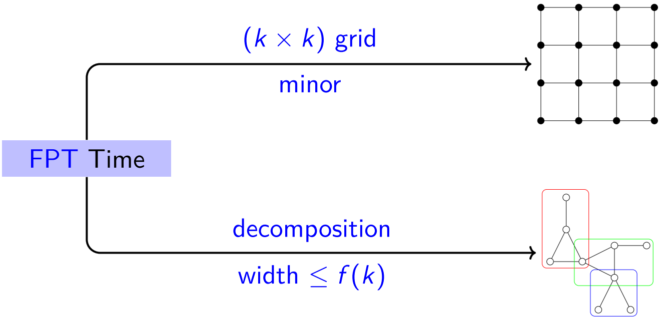

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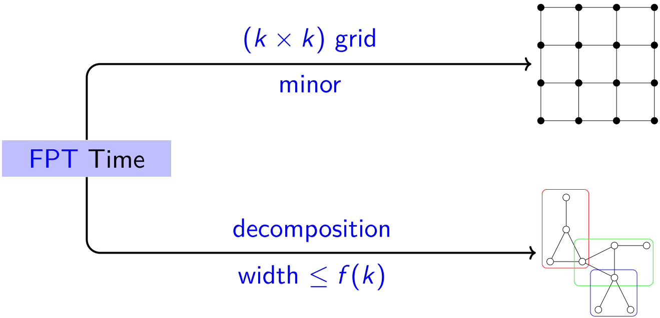

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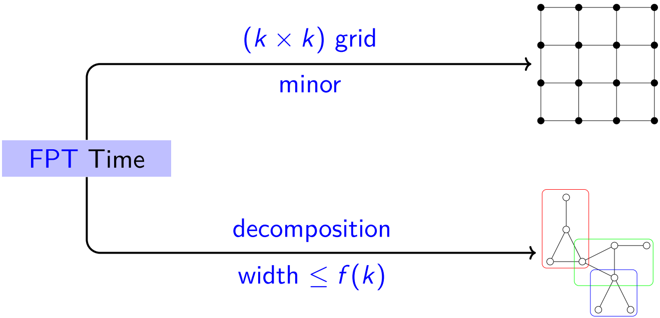

Conjecture: Directed version

Conjectured independently by

- Reed (1999).
- Johnson, Robertson, Seymour, and Thomas (2001).

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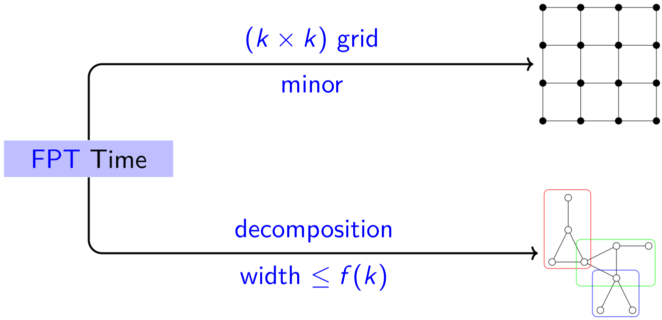

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Input: Graph G, integer k.
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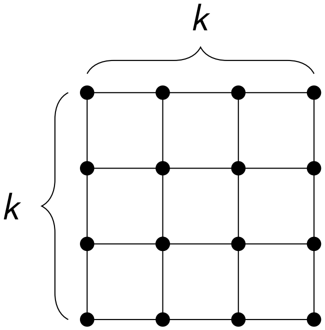
Next part based in:

M. Cygan, F. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk and S. Saurabh.
Parameterized Algorithms.
Springer, 2015.

A look at Bidimensionality

What's in a $(k \times k)$-grid?



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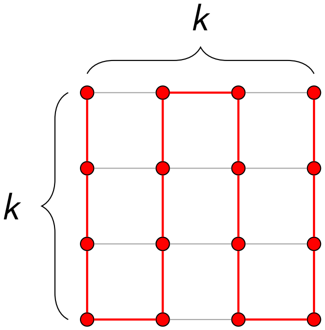

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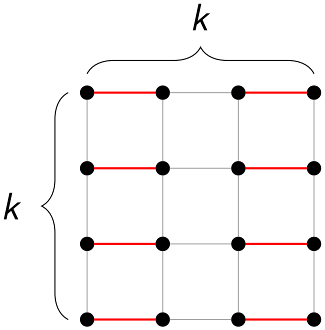

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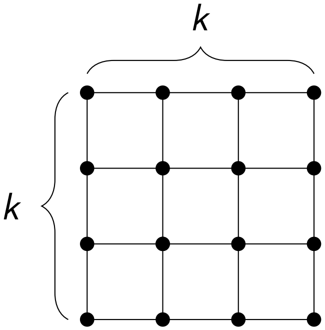

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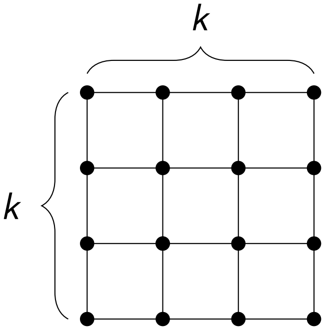

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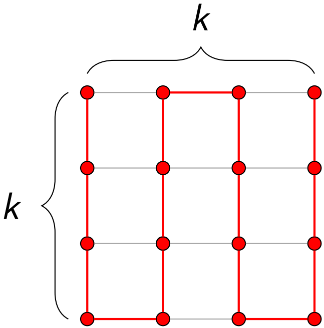

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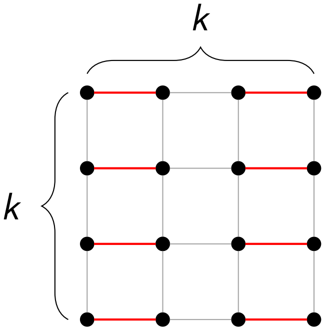

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Theorem (Planar excluded grid theorem)

Planar G + $\mathrm{tw}(G) \geq 9t/2$ *then G contains $(t \times t)$-grid minor.*
$\triangleright$ *Grid or decomposition found in $\mathcal{O}(n^2)$ time.*

N. Robertson, P. Seymour, and R. Thomas.
Quickly excluding a planar graph.
Journal of Combinatorial Theory, Series B, 1994.

Q-P. Gu and H. Tamaki.
Improved Bounds on the planar branchwidth with respect to the largest grid minor size
Algorithmica, 2012.

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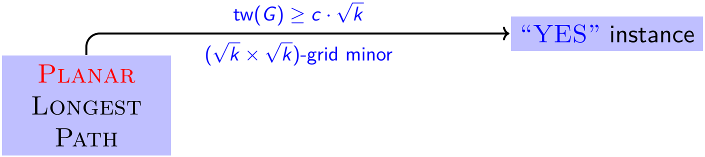

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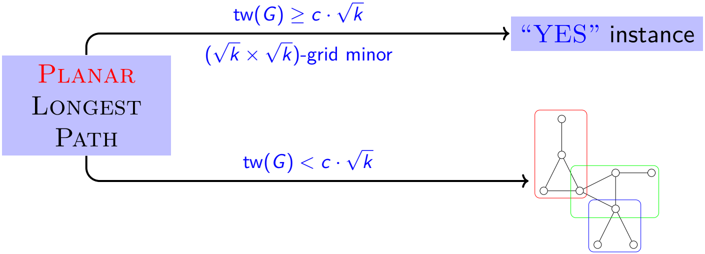

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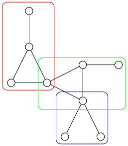
PLANAR LONGEST PATH

$\text{tw}(G) \geq c \cdot \sqrt{k}$ — $(\sqrt{k} \times \sqrt{k})$-grid minor → "YES" instance

$\text{tw}(G) < c \cdot \sqrt{k}$ →



LONGEST PATH in $2^{\mathcal{O}(\text{tw})} \cdot n$ time → PLANAR LONGEST PATH in $2^{\mathcal{O}(\sqrt{k})} \cdot n^{\mathcal{O}(1)}$ time

- Subexponential algorithms for planar Steiner Tree, Feedback Vertex Set, Longes Path, Vertex Cover, Dominating Set...

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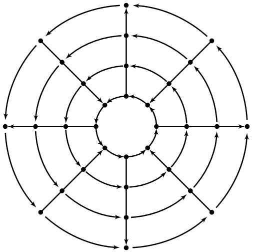
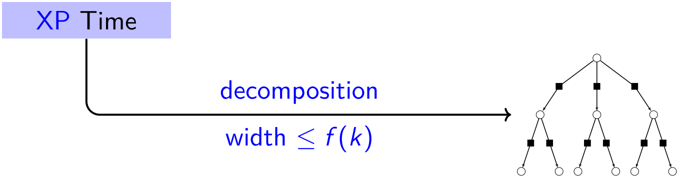


Cylindrical grid of order 4.

Directed Grid Theorem



Proof - 16 years later

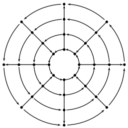
K. Kawarabayashi and S. Kreutzer.
The Directed Grid Theorem
STOC, 2015

Directed Grid Theorem

XP Time

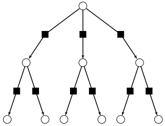
order k

cylindrical grid minor



decomposition

width $\leq f(k)$



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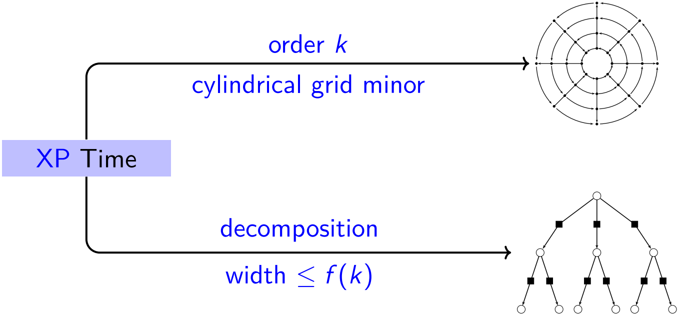
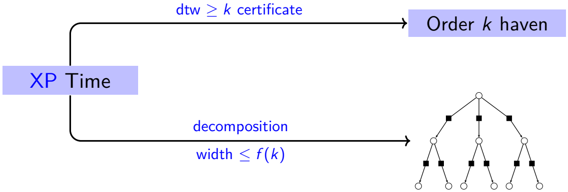


Our result

$$\text{XP} \rightarrow \text{FPT}$$

Understanding the Directed Grid Theorem



T. Johnson, N. Robertson, P. Seymour and R. Thomas.
Directed tree-width
JCTB, 2001

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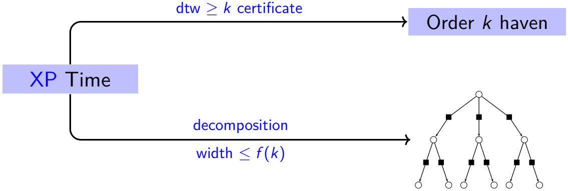

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B. Reed.
Introducing directed tree-width
ENDM, 1999

Kawarabayashi and Kreutzer.
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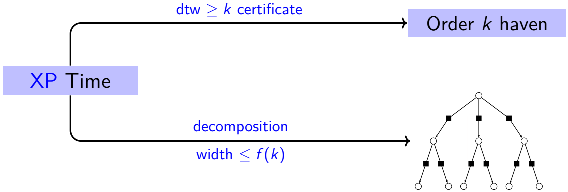

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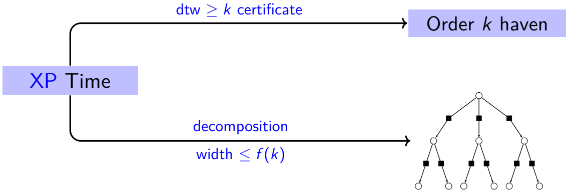

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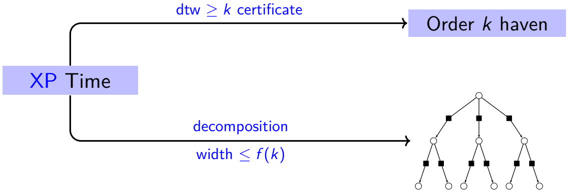

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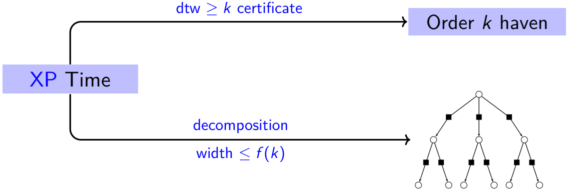

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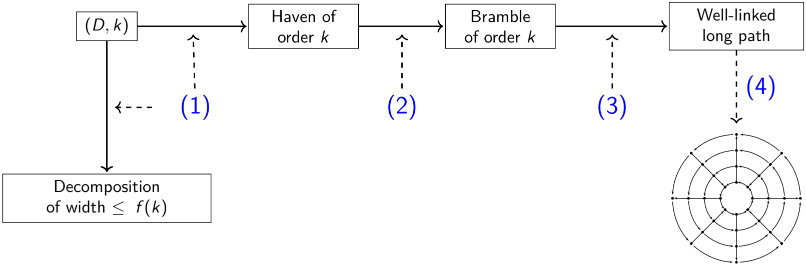
- Analyze what needs to change to achieve FPT time.

Directed Grid Theorem: constructive proof (I)



(1) Haven vs Decomposition in XP time.

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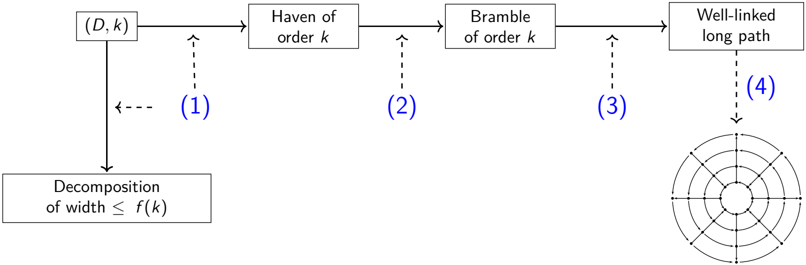

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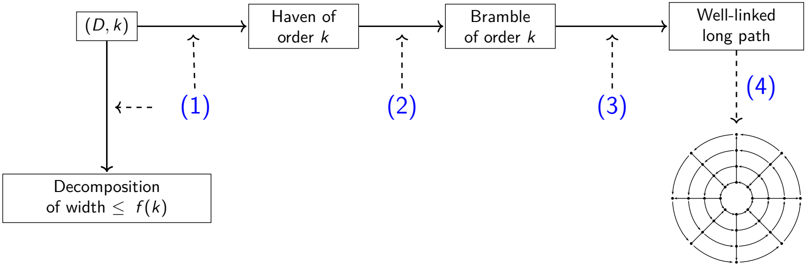

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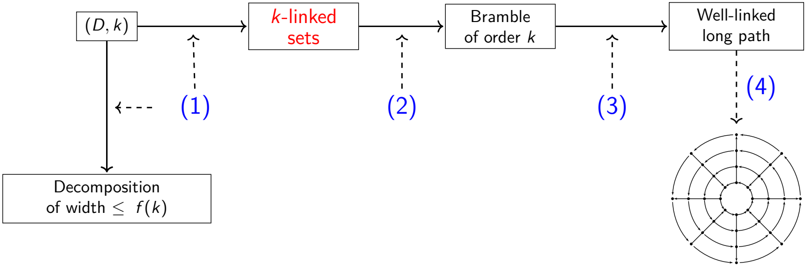

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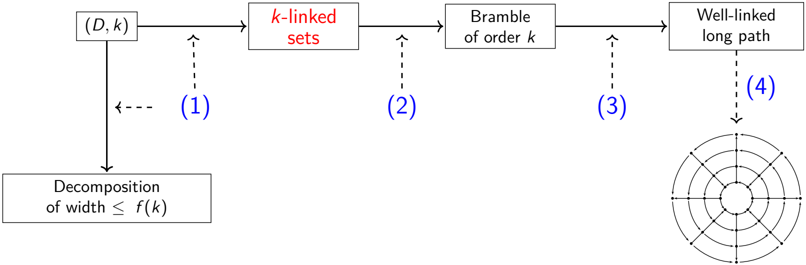

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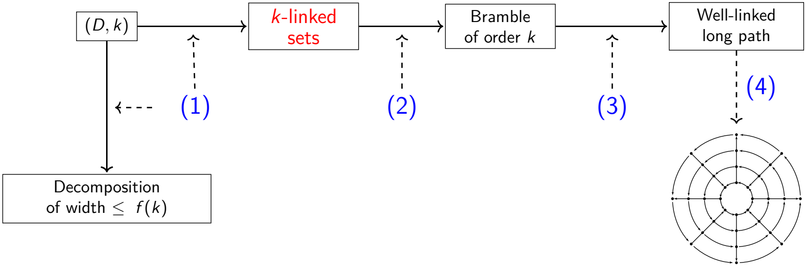

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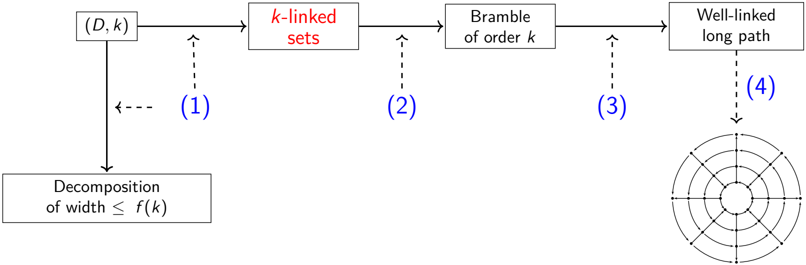

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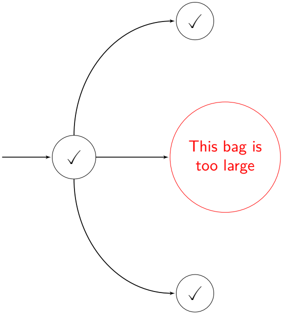

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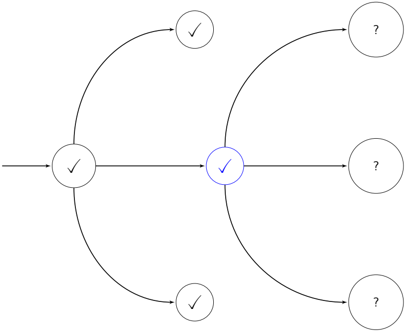

Large bags and balanced separators

- Given $T \subseteq V(D)$:

Definition (Balanced Separators, k-linked sets)

▷ $Z \subseteq V(D)$ is a *T-balanced separator* if $|T \cap V(C)| \leq \left\lfloor \frac{|T|}{2} \right\rfloor$ for every **strong** component C of $D \setminus Z$.

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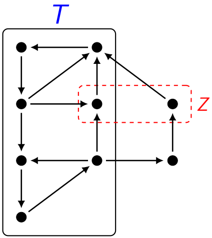

- $|T| = 7$, $\left\lfloor \frac{|T|}{2} \right\rfloor = 3$.

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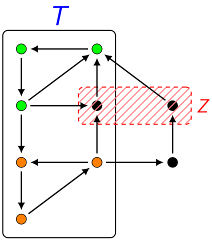

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Large bags and balanced separators

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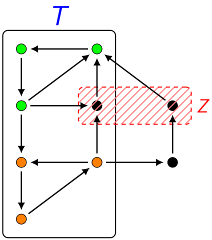

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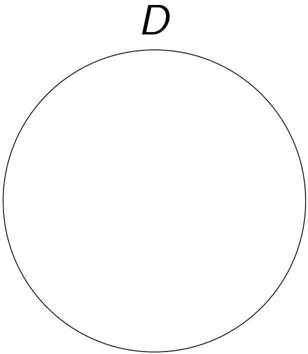
Key ingredient: $T \subseteq$ Large bag X, $|T| \leq 2k-1$: is T $(k-1)$-linked?

Finding balanced separators: XP part of JRST algorithm



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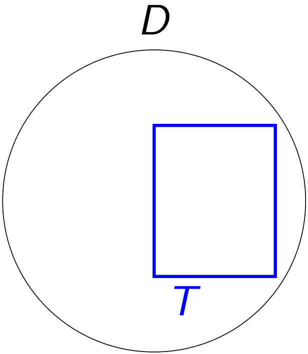

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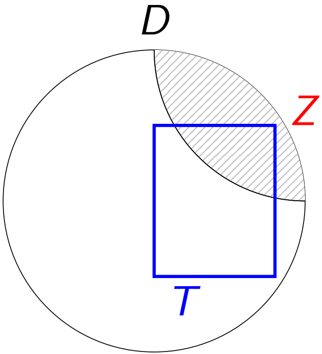

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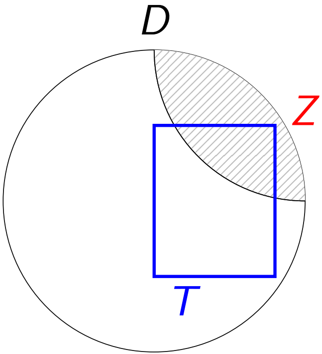

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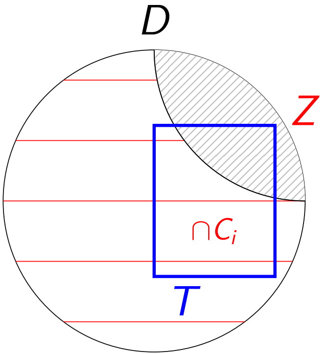

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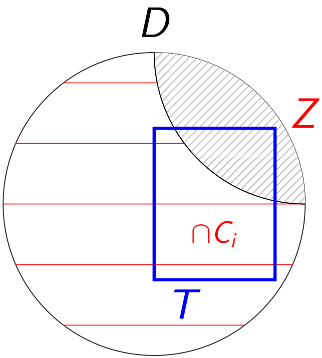

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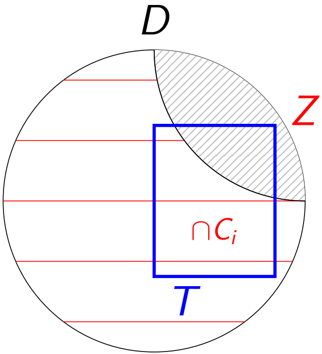

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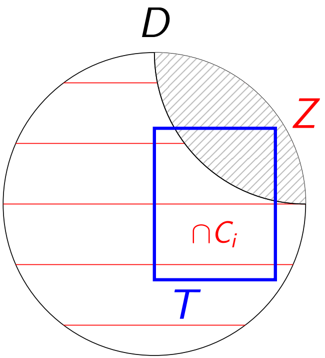

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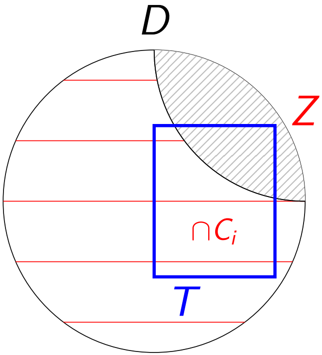

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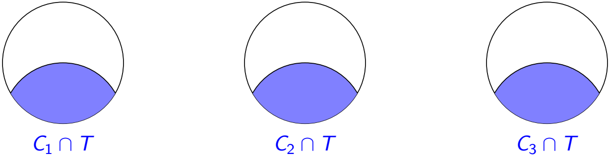

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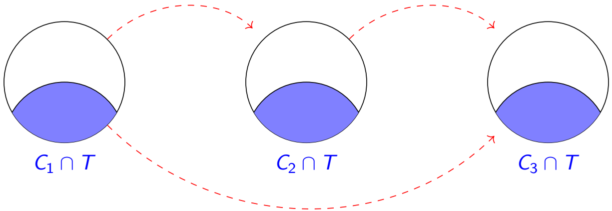

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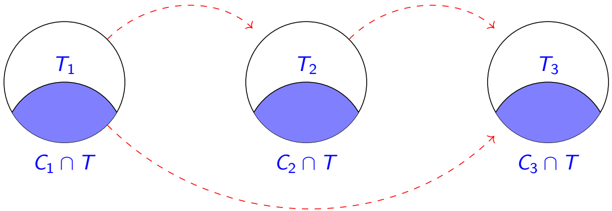

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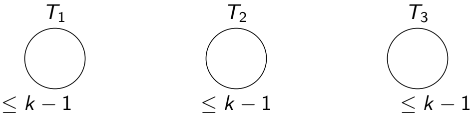

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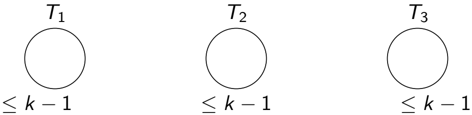

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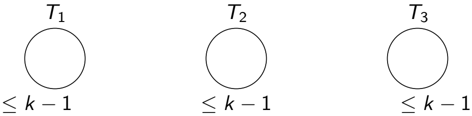

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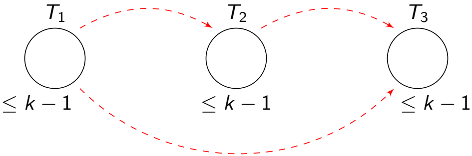

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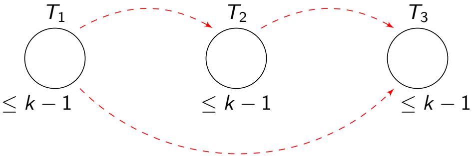
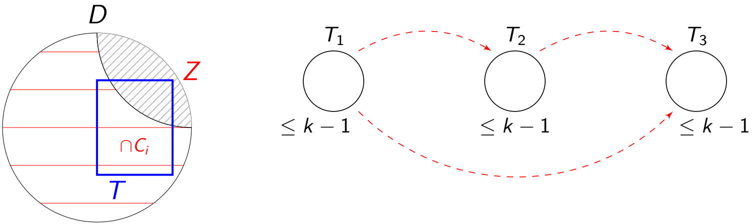


Reduces to FPT problem

R. Erbacher, T. Jaeger, N. Talele and J. Teutsch
Directed Multicut with Linearly Ordered Terminals
CoRR abs/1407.7498, 2014.

Balanced separators and $\mathcal{P}$



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∃ bal. separator $|Z| \leq k-1 \Leftrightarrow \mathcal{P}$ *positive for some partition* $T_1, \ldots, T_r$ *of* $T \setminus Z$.

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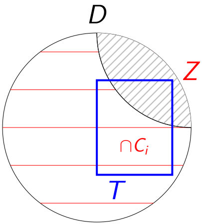

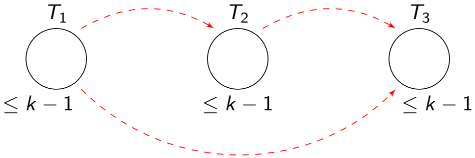

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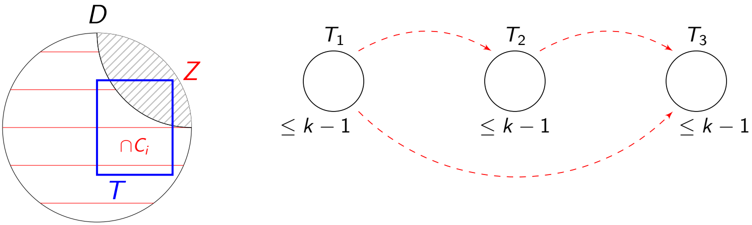

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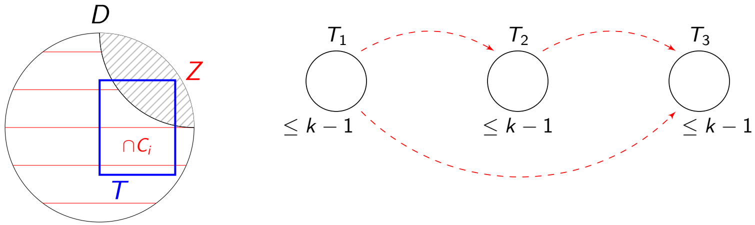

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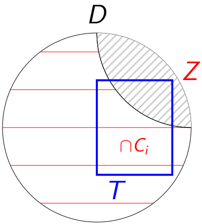

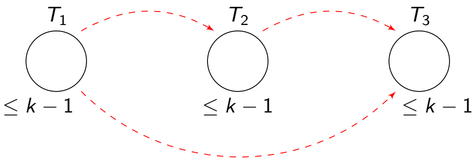

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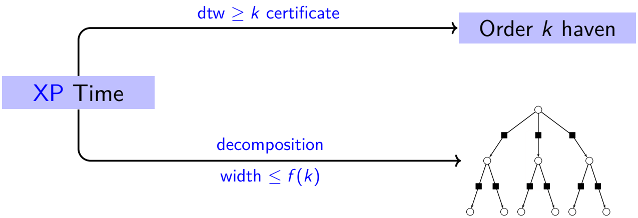
- We solve more general version, which we named "*Partitioning sets*".

k-linked sets vs Decompositions



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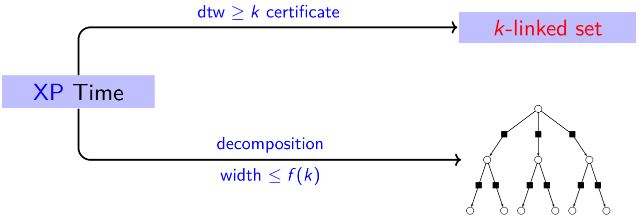

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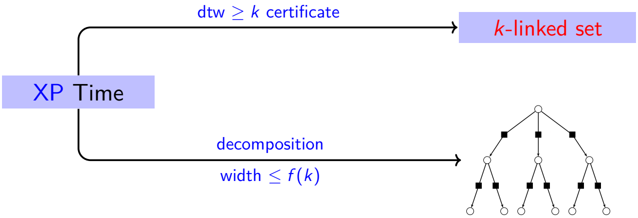

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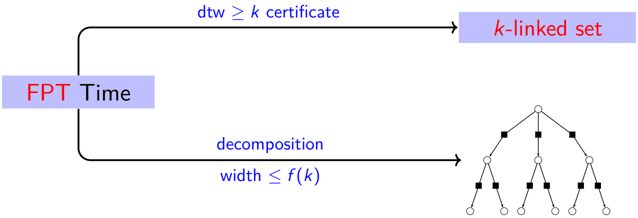

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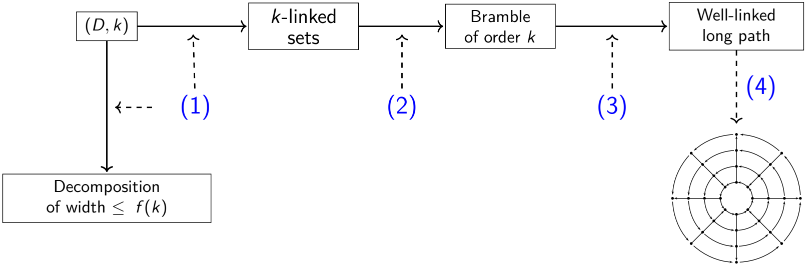
Directed Grid theorem: constructive proof (*II*)



(1) Haven vs Decomposition in XP time.

(2) Haven $\Rightarrow$ Bramble of *size* $n^{\mathcal{O}(k)}$.

(3) Bramble $\Rightarrow$ Well-linked long path: working with hitting sets (XP time).

✓(1) *k-linked sets* vs Decomposition in FPT time.

(2) *k-linked sets* $\Rightarrow$ Bramble that is *easier* to work with.

(3) Bramble $\Rightarrow$ Well-linked long path: working with hitting sets (FPT time).

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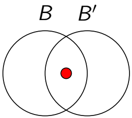

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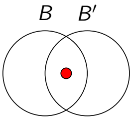

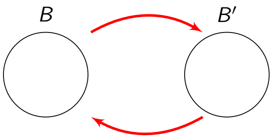

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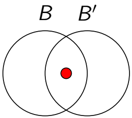

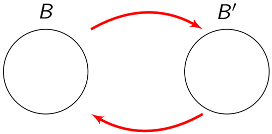

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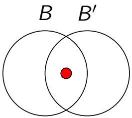

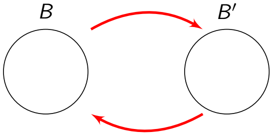

- hitting set of $\mathcal{B}$ = set of vertices touching every $B \in \mathcal{B}$.
- order of $\mathcal{B}$ = minimum size of hitting set.

Working with brambles

Bramble $\mathcal{B} = \{B_1, B_2, \ldots, B_\ell\}$.

- Naive approach to find hitting sets w. size k (assuming $|\mathcal{B}| > k$):

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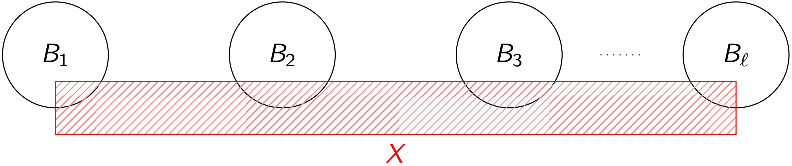

Running time: $\mathcal{O}(n^k) \cdot |\mathcal{B}|$.

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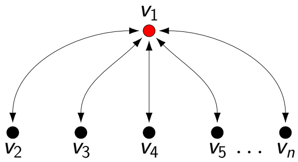

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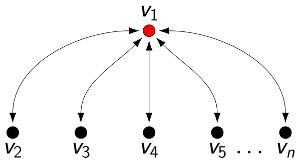

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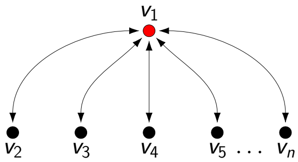

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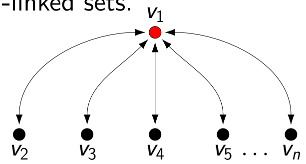
- Better bramble from k-linked sets.



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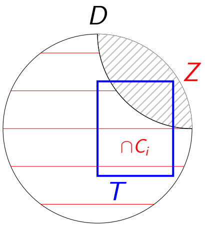
Brambles from k-linked sets



- $|T| \leq 2k-1$.
- T is $(k-1)$-linked $\Rightarrow$ no $k-1$ bal. separator

No $Z \subseteq V(D)$ with $|Z| \leq k-1$ satisfies:

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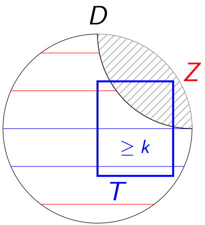
Brambles from k-linked sets



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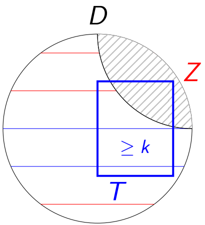

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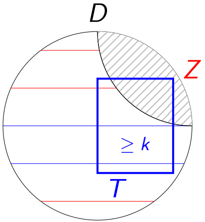

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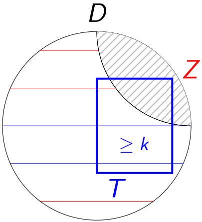

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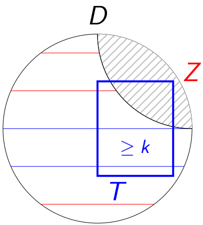

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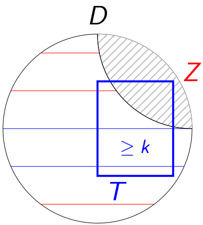

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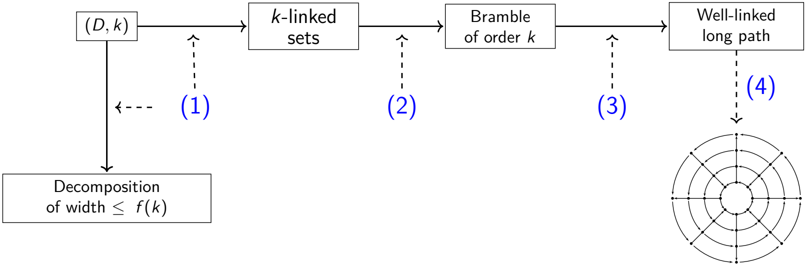
- For $\mathcal{B}' \subseteq \mathcal{B}_T$, is order$(\mathcal{B}') \leq q$?
  - Solvable through T-*partitioning sets* $\Rightarrow$ FPT time (appropriate choices of $\mathcal{B}'$).
    - ⋆ (*Partitioning sets* generalize *balanced separators*).

Directed Grid Theorem: constructive proof (*III*)



(1) Haven vs Decomposition in XP time.

(2) Haven $\Rightarrow$ Bramble of *size* $n^{\mathcal{O}(k)}$.

(3) Bramble $\Rightarrow$ Well-linked long path: working with hitting sets (XP time).

✓(1) *k-linked sets* vs Decomposition in FPT time.

✓(2) *k-linked sets* $\Rightarrow$ Bramble that is *easier* to work with.

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Step 1 + Step 2 $\Longrightarrow$ Well-linked set in P.

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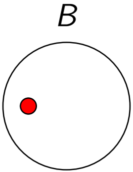

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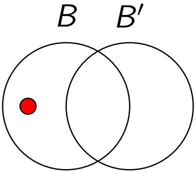

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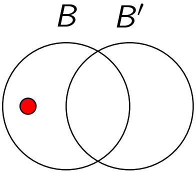

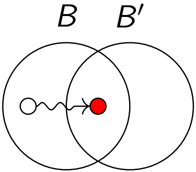

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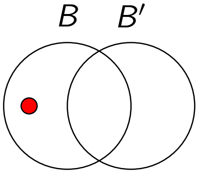

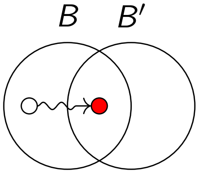

Key ingredient 1

Is Z a hitting set of $\mathcal{B}_T$?

EASY: Does $D \setminus Z$ contains a **strong** component intersecting k vertices of T?

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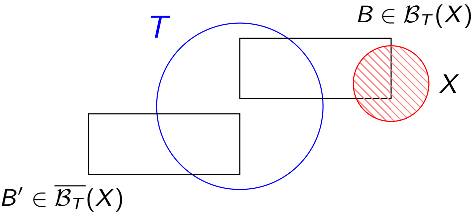

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Not easy. However,

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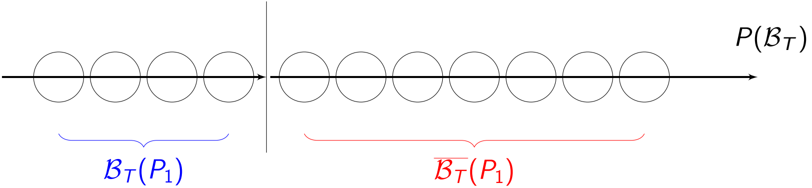

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$$\textit{Given } P \textit{ hitting } \mathcal{B}_T, \text{order}(\mathcal{B}_T) \geq (k+1)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) - 1$$

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  - Used to construct highly-connected *system of paths*.

K. Kawarabayashi and S. Kreutzer.
The Directed Grid Theorem
STOC, 2015

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M. Hatzel, K. Kawarabayashi, and S. Kreutzer.
Polynomial Planar Directed Grid Theorem
SODA'19

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- Is there an interesting problem can be shown FPT using this?

Applications

- H, **strongly** *connected*, has Erdős-Pósa property $\Leftrightarrow$ H is *butterfly* (or topological) minor of a cylindrical grid.

S. Amiri, K. Kawarabayashi, S. Kreutzer and P. Wollan.
The Erdős-Pósa property for directed graphs
CoRR abs/1603.02504, 2016

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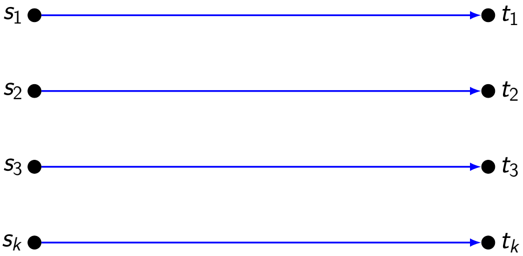

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A. C. Giannopoulou, K. Kawarabayashi, S. Kreutzer, and O. Kwon.
The Directed Flat Wall Theorem
SODA'20

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M. Oliveira.
An algorithmic metatheorem for directed treewidth
DAM'16

Brambles with constant congestion

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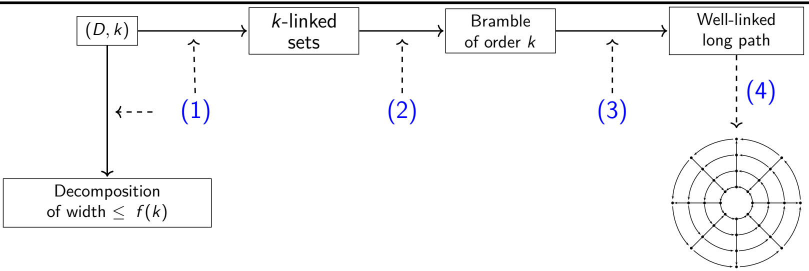
  - *Question:* can this bramble be constructed in FPT time?

THANKS!



(1) Haven vs Decomposition in XP time.

(2) Haven $\Rightarrow$ Bramble of *size* $n^{\mathcal{O}(k)}$.

(3) Bramble $\Rightarrow$ Well-linked long path: working with hitting sets (XP time).

✓(1) *k-linked sets* vs Decomposition in FPT time.

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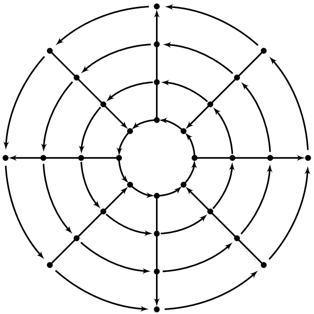
Cylindrical Grid



Cylindrical Grid of order k

- k cycles, same direction.
- $2k$ alternating paths.

Partitioning sets

- Given $T \subseteq V(D)$:

Definition ((T, r)-Partitioning Sets)

▷ $Z \subseteq V(D)$ is a *(T, r)-partitioning set* if $|T \cap V(C)| \leq r$ for every **strong** component C of $D \setminus Z$.

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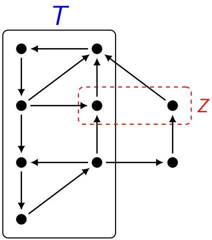

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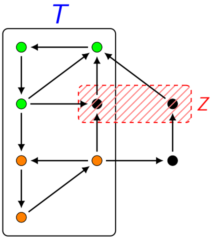

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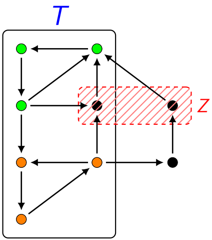

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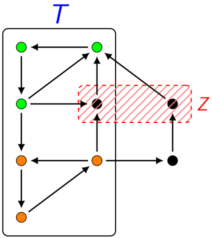

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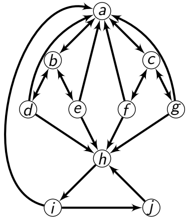
- Question: FPT algorithm when parameterized by $|Z|$?
  - $T = V(D), r = 0 \implies$ FEEDBACK VERTEX SET.

Directed tree-width-I



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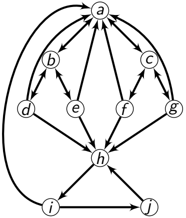

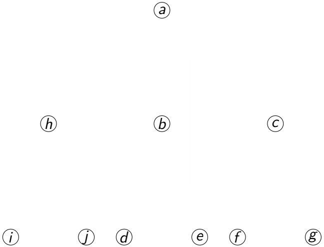

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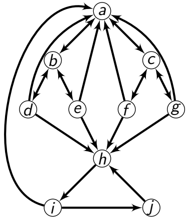

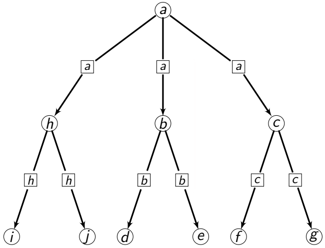

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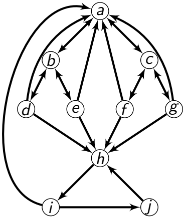

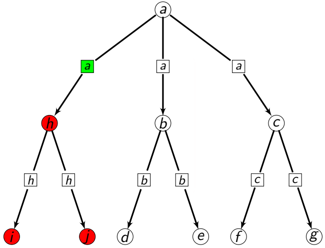

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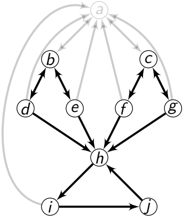

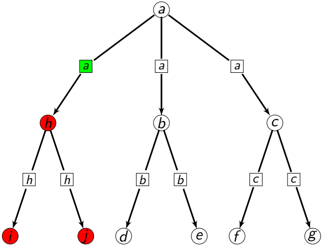

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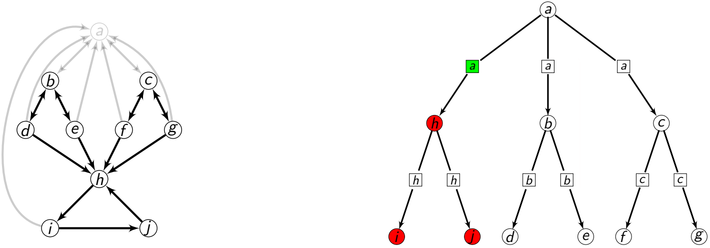

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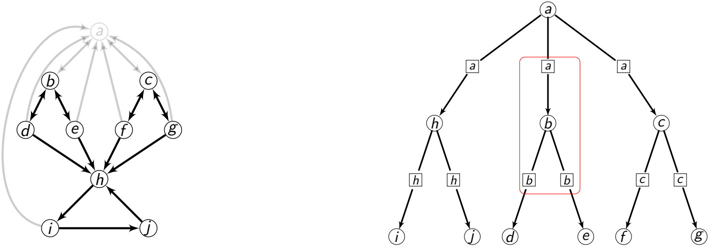

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- width = size of largest set of "bag" + adjacent "guards".

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- Estimation to how close a directed graph is to a DAG.

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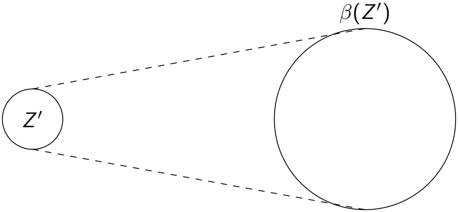

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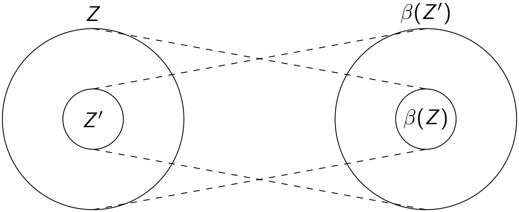

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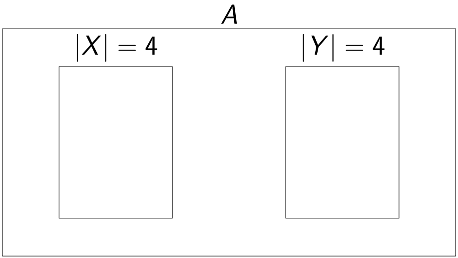

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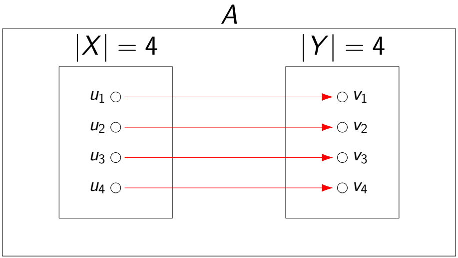

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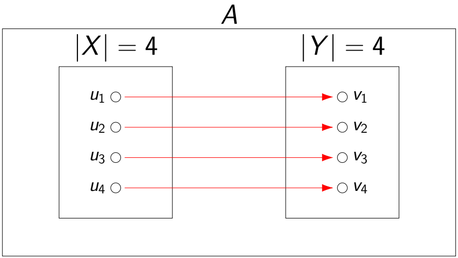

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