Edge-Disjoint Branchings in Temporal Graphs

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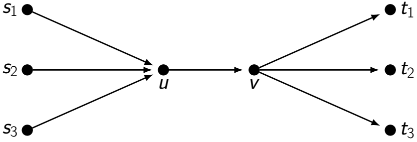
Temporal Graphs

- Digraph that changes with time.



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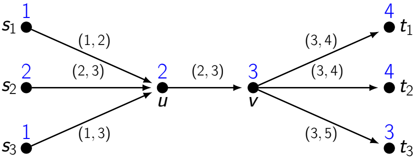

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- add *vertex times*.
- (*departure*, *arrival*) times for the edges.

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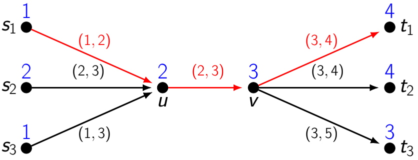

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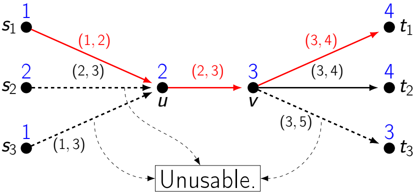

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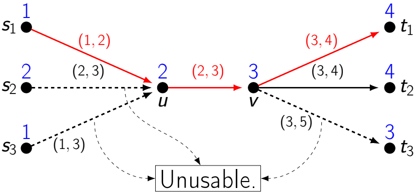

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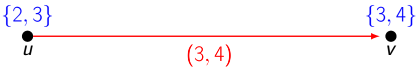
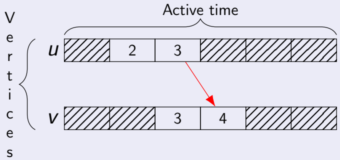
Summary

- *Temporal graph* $\mathcal{G}$ = base static digraph D plus
  - *Activity times* for the vertices, *departure* and *arrival* times for the edges.
  - v is *permanent* if v is always *active*.

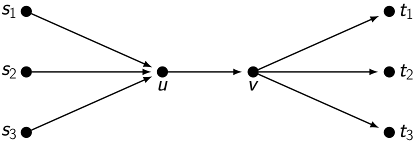
Alternative view

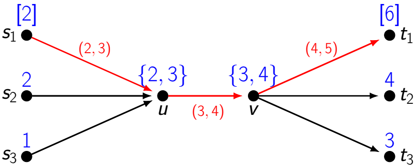


Temporal expansion



Using $[k] = \{1, \ldots, k\}$.





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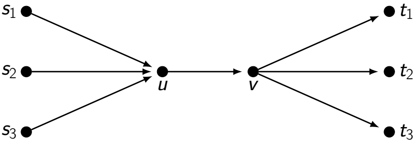

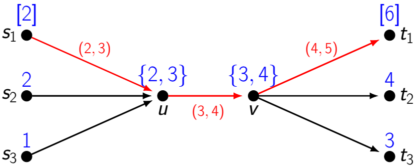

- Use red edges at correct time to construct $(s_1, 1) \rightarrow (t_1, 6)$ *temporal path*.

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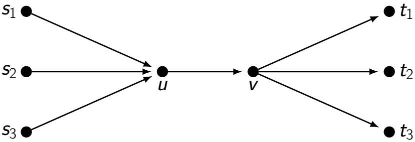

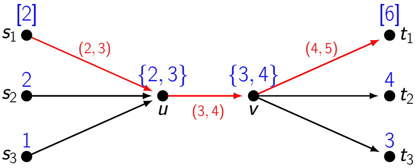

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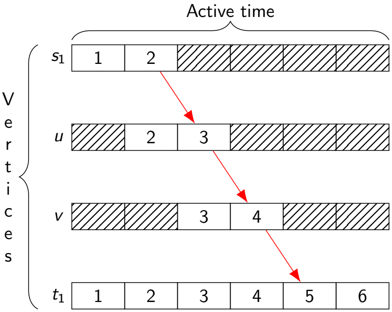

- $(s_1, 1)$ wait $(s_2, 2) \rightarrow (u, 3) \rightarrow (v, 4) \rightarrow (t_1, 5)$ wait $(t_1, 6)$.

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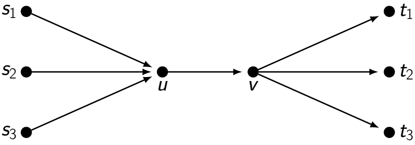

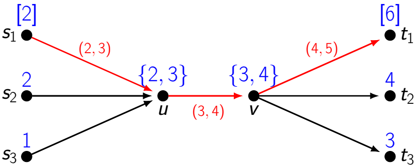

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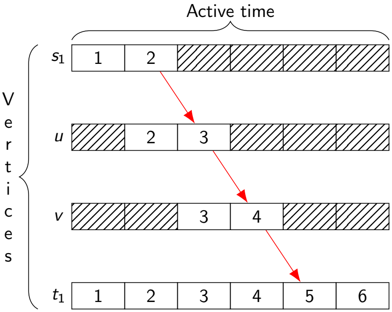
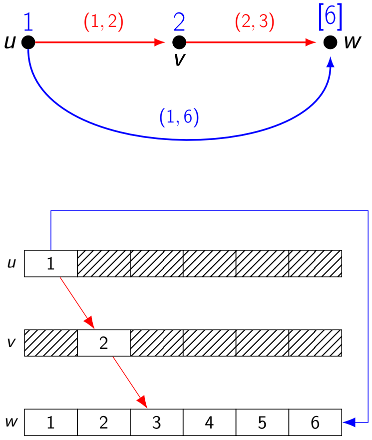


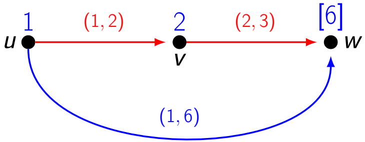
- $(s_1, 1)$ wait $(s_2, 2) \rightarrow (u, 3) \rightarrow (v, 4) \rightarrow (t_1, 5)$ wait $(t_1, 6)$.
- Lifetime $= 6$, t_1 is *permanent*.

Understanding temporal paths

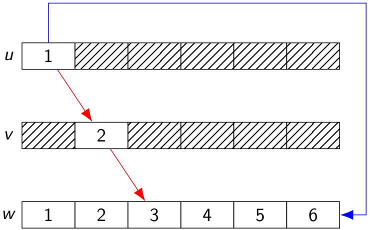


- $\exists$ two $u \to w$ temporal paths.

Understanding temporal paths



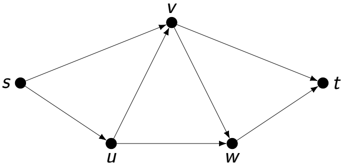
- ∃ two $u \rightarrow w$ temporal paths.

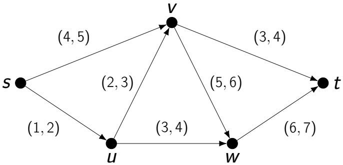


- *Blue path* has min #edges.
- *Red path* has *shortest arrival time* $(u, 1) \rightarrow (w_3)$.

Understanding temporal paths II

- *Menger's Theorem*: Vertex version depends on *cut interpretation*.





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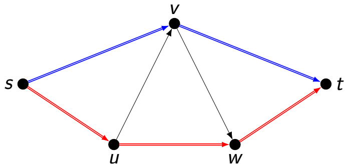

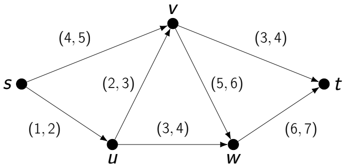

- Two internally disjoint $s \to t$ paths.
- $|\text{Separator}| = 2$.

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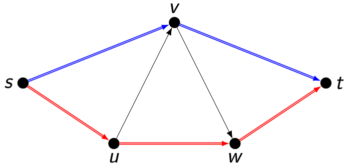

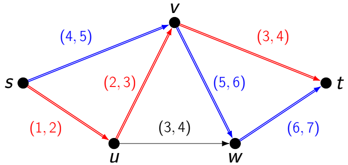

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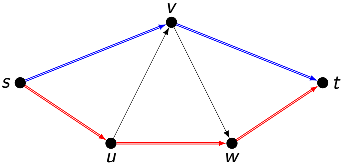

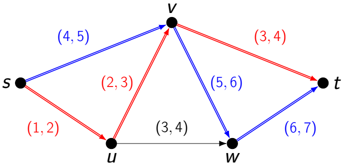

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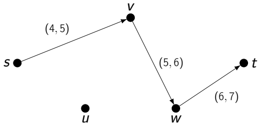
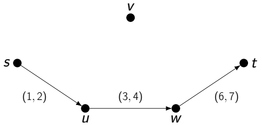
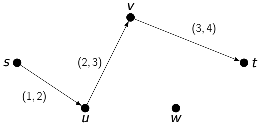
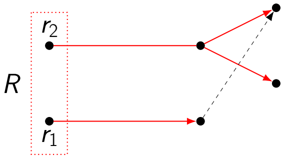






Disjoint branchings

- *Spanning branching with root R*: path from $R \rightarrow$ every $v \notin R$.



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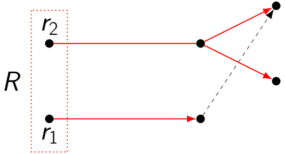
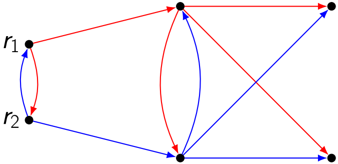


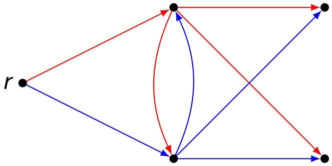
- *Disjoint* spanning branchings with roots r_1, r_2.



Disjoint branchings II

Theorem (Edmonds, 1973)

A digraph D has k edge-disjoint branchings rooted at r $\iff$ $d^{-}(X) \geq k$ for all $X \subseteq V(D) - r$.



Jack Edmonds.
Edge-disjoint branchings.
Combinatorial Algorithms, 1973.

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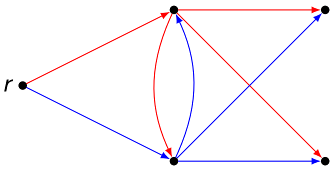

- Characterization $\Longrightarrow$ polynomial time algorithm.
- Particular case of flow problem that is, in general, hard.

Jørgen Bang-Jensen and Stéphane Bessy.
(Arc-)disjoint flows in networks.
Theoretical Computer Science, 2014.

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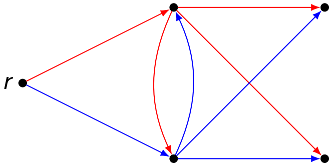


- Characterization $\implies$ polynomial time algorithm.
- Particular case of flow problem that is, in general, hard.
- Does not hold for temporal graphs.
  - One of many cases we consider.

David Kempe, Jon Kleinberg and Amit Kumar.
Connectivity and inference problems for temporal networks.
Proceedings of the 32nd annual ACM Symposium on Theory of Computing (STOC), 2000.

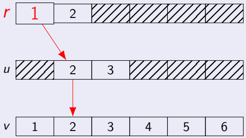
Temporal branchings I

- *Vertex* spanning VS *Temporal* spanning.

Temporal branchings I

- *Vertex* spanning VS *Temporal* spanning.
  - Span every vertex VS

Vertex spanning

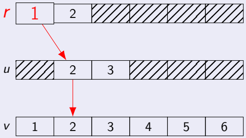


- Spans all vertices.

Temporal branchings I

- *Vertex* spanning VS *Temporal* spanning.
  - Span every vertex VS Span every vertex at every time.

Vertex spanning



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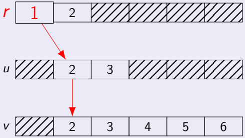
Temporal spanning



- Spans all *temporal* vertices.

Temporal branchings II

- *Edge*-disjoint branchings VS *Temporal*-disjoint branchings.

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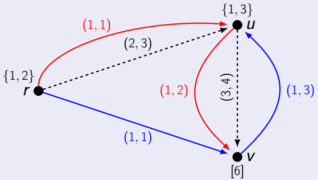
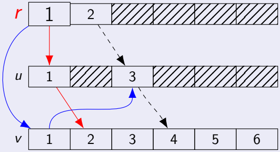
- *Edge*-disjoint branchings VS *Temporal*-disjoint branchings.
  - Not using same *base* edge VS

Edge-disjoint





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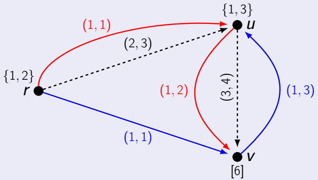

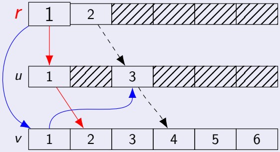

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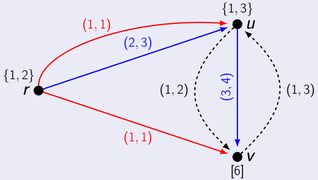
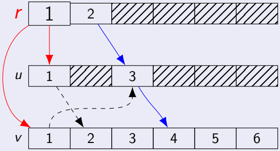
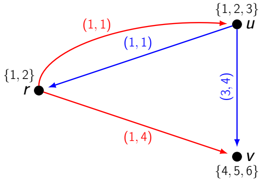
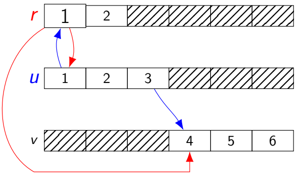




Finding k Temporal-disjoint + temporal spanning

- Temporal graph $\mathcal{G}$ with base digraph D.





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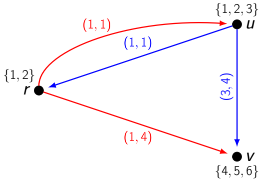

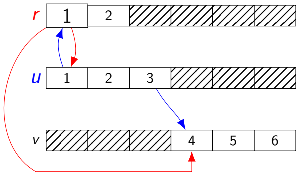

- Input on the left adapted to digraph on the right.
- Polynomial time algorithm by Edmonds'.

Finding k Edge-disjoint + temporal spanning

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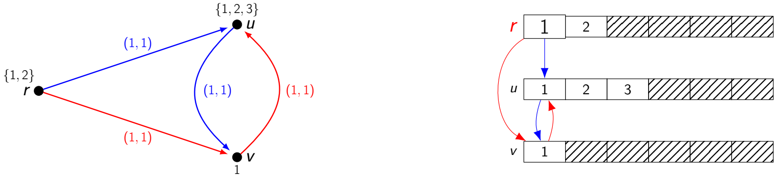

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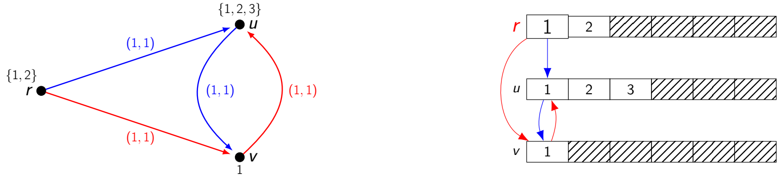
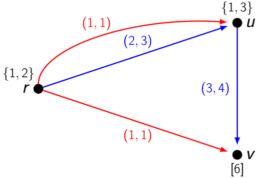
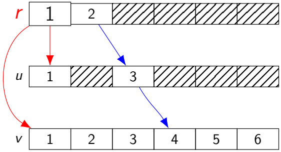


- NP-complete even if D is in-star and each snapshot has constant size.
- NP-complete if $\mathcal{G}$ has lifetime ≥ 3.
- Solvable in Polynomial time if all vertices are permanent.

Temporal-disjoint + vertex spanning

- Temporal graph $\mathcal{G}$ with base digraph D.





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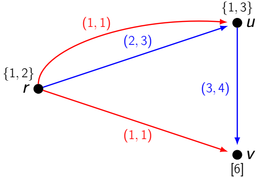

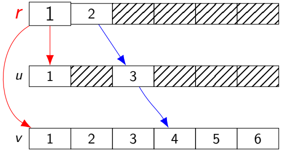

- NP-complete even if
  - D is a DAG,
  - $\mathcal{G}$ has lifetime 2, and
  - all vertices are permanent

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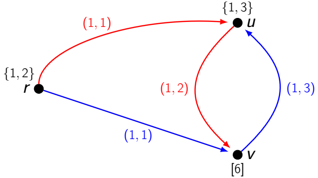
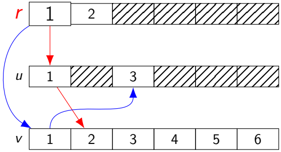
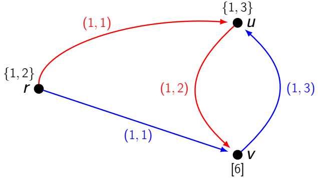
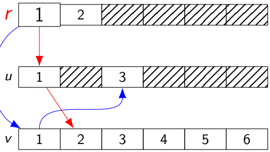




Edge-disjoint + vertex spanning

- Temporal graph $\mathcal{G}$ with base digraph D.





- NP-complete even if
  - D is a DAG,
  - $\mathcal{G}$ has lifetime 2, and
  - all vertices are permanent

Open questions

- Parameterized complexity: polynomial time algorithm for
  - fixed lifetime? (except edge-disjoint, temporal spanning).
  - fixed treewidth? (except vertex spanning variants).
- Algorithms matching computational lower bounds under ETH?

Summary

- We consider 4 definitions for *temporal branchings*.
- Edmonds' characterization not true in general.

	not permanent vertices		permanent vertices	
	edge-disjoint	t-edge-disjoint	edge-disjoint	t-edge-disjoint
temporal-spanning	NP-c[1]	Poly*	Poly	Poly
vertex-spanning	NP-c[2]	NP-c[2]	NP-c[2]	NP-c[2]

Our results. Vertices are permanent if they are always active.

* Edmonds' characterization for temporal expansion.

1 Even if D is an in-star and each snapshot has constant size; or if $\mathcal{G}$ has lifetime ≥ 3

2 Even if D is a DAG, $\mathcal{G}$ has lifetime ≥ 2, and all vertices are permanent.